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NONLINEAR DYNAMICS & THE INTERMITTANCY ROUTE TO CHAOS

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Introduction to Dynamical Systems

1.1 Coupled Differential Equations

The study of nonlinear dynamics originates in the study of ordinary differential equations. Some characteristics of ordinary differential equations (ODEs) are shared by partial differential equations (PDEs) as well, so it is logical to introduce this simpler concept first.

One mathematically interesting type of ODE is that of a **coupled** ODE. The word "coupled", indicates that the dynamics of one variable depend on the dynamics of another at any given moment in time. The equation for a damped harmonic oscillator, for example, may be written as

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \tag{1.1}$$

The mathematics of differential equations (no proof of this will be shown) dictates that **any N-th order single ODE may be re-written as N separate first-order ODEs**.

Given the substitutions of $x_1 = x$, $x_2 = \dot{x}$, $\dot{x_1} = x_2$, $\dot{x_2} = \ddot{x}$ and as the above is only second order, we note the two equations below, and claim they are a partitioned version of the original equation representing the spatial oscillation of a damped harmonic oscillator in time:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1$$

Although these are two separate equations, the physical properties are identical to Equation (1.1). The system in this case is also said to be **linear**, as terms on the right hand side of the equations are to first power only. Any deviation from this rule would deem the system as **nonlinear**. Steven Strogatz writes[1] that typical nonlinear terms include products, powers and embedded functions of x_i (the variables in question) such as x_1x_2, $(x_1)^3$ or $cos(x_3)$.

An example of a **nonlinear system** would be the true, nonlinearised model of a swinging pendulum given by

$$\ddot{x} + \frac{g}{L}sin(x) = 0$$

[1] *Nonlinear Dynamics and Chaos, pg. 6*

Due to this being a 2nd order ODE, the equivalent partitioned, coupled system, via the aforementioned method, is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{L} sin(x_1)$$

Here, the dynamical dependence on the *sin(x)* function is what makes the system nonlinear in nature.

Time could be substituted with another variable (x_3, for example) and can be introduced into a system with two other spatial equations (as say, $\dot{x_3}$). For all intents and purposes, evolution of time is constant and is usually written as $\dot{x_3} = 1$, such that x_3=constant. This convention opens up the exotic possibility for a system to work with nonlinear time, if such a scenario demanded such a concept.

When working with coupled systems, it is necessary to consider the **phase space** of the system. For two coupled equations, we work with a **phase plane**, which is a plane that illustrates how values of the corresponding solutions (x_1 and x_2) from a system interact with one another. Figure (5.1) is an example of what a typical phase portrait might look like.

The true **difficulty** in studying nonlinear systems comes from the fact that in treating aggregated and complex problems in math and physics, only so much may be done analytically. Undergraduate-level physics and mathematics problems are spoiled with closed-form (albeit messy) solutions to most if not all problems. Non-linear problems often require the use of numerical methods to approximate solutions. Although not going to be covered in this thesis, the algorithms used to perform such calculations are also worth studying in and of themselves due to the prevalence of computer approximation errors which have been known to jeopardize results, as in the Euler method. This issue can also impede the exacting efforts of even the most precise algorithms.

Data obtained from numerical solutions can also be verified via experiment (in the case of physical systems) to verify the authenticity of an algorithm and research at various universities has been moving forward to do just that.

1.2 Fixed Points and Stability Analysis

Fixed points are a crucial aspect of dynamical systems. These points describe where the system is no longer dynamic, almost like an oasis in a vast, uncertain, differential-equation desert. Many researchers seek this oasis of certainty within their research, especially when describing nonlinear, chaotic systems.

Fixed points are traditionally found using the following convention

$$\dot{x}_1 = 0$$

$$\dot{x}_2 = 0$$

$$\vdots$$

$$\dot{x}_n = 0$$

This means one must look for point(s) with no rate of change in all applicable variables. An example is that of velocity. We recall from physics that velocity is simply the *time derivative* of displacement. If we define x_1 as displacement, $\dot{x}_1$ is its respective velocity. Upon contemplation of our aforementioned example of the damped harmonic oscillator, we can make sense of this point:

$$\dot{x}_1 = x_2 = 0$$

$$\dot{x}_2 = -\frac{b}{m}x_2 - \frac{k}{m}x_1 = 0$$

When we solve this system, we note that the first equation tells us that $x_2 = 0$. Since the two ODEs are coupled, this information must combine with the second equation which results in

$$-\frac{b}{m}x_2 = +\frac{k}{m}x_1$$

Since we know that $x_2 = 0$, this forces x_1 and x_2 both to be equal to 0. In terms of the phase plane, this represents the point $(x_1, x_2) = (0, 0)$. **The point $(0, 0)$ is then a fixed point of the continuous system**. As we will see very soon, a fixed point, also known as an equilibrium point of a system, can also be classified as either a **stable fixed point** or an **unstable fixed point**. If unstable, the dynamics of the ODEs are then halted at this fixed point until a sufficient perturbation comes along to change its state.

This method may be reproduced for more complicated systems to arrive at fixed points which depend on *constants and parameters or even other variables.* Some systems may have one fixed point whereas others may have several. The amount depends on the level of complication of the initial dynamical system.

The behaviour of solutions *independent of location* is also of particular interest. Eigenvalues of a differential equation's **Jacobian matrix** can tell us valuable information about a system's long-term behaviour. Indeed, the eigenvalues single-handedly determine the stability of the fixed points for the entire system. Using the damped harmonic oscillator once again as a template:

$$J = \begin{pmatrix} \frac{\partial \dot{x_1}}{\partial x_1} & \frac{\partial \dot{x_1}}{\partial x_2} \\ \frac{\partial \dot{x_2}}{\partial x_1} & \frac{\partial \dot{x_2}}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}$$

$$|J - I\lambda| = 0 \rightarrow \begin{vmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{b}{m} - \lambda \end{vmatrix} = 0$$

$$-\lambda(-\lambda - \frac{b}{m}) + \frac{k}{m} = 0$$

$$m\lambda^2 + b\lambda + k = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4km}}{2m}$$

The exact value for λ depends on the relative magnitudes of the parameters in question (here b, k and m). Keeping in mind that this system represents the dynamics of a damped harmonic oscillator, we recall that all of these are physical quantities. Thus, manipulating the values of the ensemble also has a physical consequence.

One downside of the Jacobian matrix is that due to it being a linearization of the original set of differential equations, **it is only able to determine behaviour *around a fixed point***. This means that for each fixed point, we must have a *separate Jacobian matrix* for analysis.

1.2.1 Stability Criterion for Continuous Systems

To determine whether a fixed point is **stable or unstable**, we need only to look at the sign of the eigenvalue (here, denoted by the Greek letter λ) from the Jacobian matrix in question. If $\lambda > 0$, then the the eigenvalue is associated with an **unstable fixed point**. If $\lambda < 0$, the eigenvalue is associated with a **stable fixed point**.

If working in a two dimensional system with two distinct eigenvalues, and if $\lambda_1 < 0$, but $\lambda_2 > 0$, then we call this point a **saddle point**. We may use the eigenvectors associated with given eigenvalues to show that this concept of a saddle point is completely analogous to the saddle points of 3-D surfaces encountered in an undergraduate multivariable calculus course[2].

[2]Saddle points are typically visualized as a vector field in the phase plane having solutions being

We use the example from the previous section to state that in this second order harmonic system, there are two eigenvalues:

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4km}}{2m}$$

$$\lambda_2 = \frac{-b - \sqrt{b^2 - 4km}}{2m}$$

We point out that provided $\sqrt{b^2 - 4km}$ is a positive, real quantity, λ_2 will always be negative. This makes λ_2 a stable eigenvalue, but what of λ_1? The answer lies within the parameters we've set in the problem.

- If $\sqrt{b^2 - 4km} > b$, λ_1 will be positive. (λ_1 is **UNSTABLE**)
- If $\sqrt{b^2 - 4km} < b$, λ_1 will be negative. (λ_1 is **STABLE**)

Since we are looking at either both eigenvalues being negative, indicative of a **fully stable fixed point**, or one negative and one positive, **indicative of a saddle point**, it is crucial to look at *all* the possible eigenvalues for a particular Jacobian matrix.

Although avoided for simplicity, it is often the case that λ is actually a function of variables, i.e. $\lambda = \lambda(x_1, x_2, \theta, r, h)$. Due to this phenomenon, **the coordinates in phase space may determine whether that point is stable or unstable**. This is more dynamic than the example of simple harmonic oscillator (since its Jacobian had no dependence on system coordinates). It is exactly this dependence on either spatial

attractive in one axis and repulsive in another axis or temporal coordinates that makes dynamical systems so intriguing. Oscillating between stable and unstable states can clearly have a profound effect on any physical system that can be described using systems of differential equations.

1.3 Bifurcation Diagrams

Strogatz defines a **bifurcation** as “a qualitative change in the dynamics of a system”. Due to the fact that some coupled ODE systems have parameters which may change in the lifetime of the system, it’s natural that signs and values of variables may change. It is this change which causes fixed points to be created or destroyed while maintaining the overall symmetry or asymmetry of a system. In addition, the variations of values of parameters in a system may even change the stability of fixed points, causing a sudden shift in the shape of the phase plane or the flow of a $\dot{x}$ vs. x plot (phase plane portrait).

One example of such a system is

$$\dot{x} = \mu + x^2 \tag{1.2}$$

where μ is known as a **bifurcation parameter**.

Now let’s suppose $\mu = 0$, which would make $\dot{x} = x^2$, which is a parabola in the phase plane. By visually assessing the phase plane, we may also find fixed points geometrically. Looking at $\dot{x}$ vs. x, we denote fixed points as any coordinate location where the curve in question meets the x-axis (at $\dot{x}$=0). This occurs at the origin again, at $(0, 0)$. **One definite fixed point is found**. Any other value of μ and the behaviour changes...

- For $\mu < 0$, the parabola dips below the x-axis, bifurcating into a regime where **two fixed points** exist.
- For $\mu > 0$, the parabola dips above the x-axis, bifurcating into a regime where **no fixed points** exists.

This is an example of a **saddle-node bifurcation**. Given the various shaped curves a system may trace on the phase plane, there are different kinds of bifurcations a system can undergo. Examples of various bifucations are illustrated in the Appendix of this paper. Figure 5.3 shows an axis system of x vs. μ, which shows the progression of fixed point evolution at a sweep of bifurcation parameter values . Stable branches of fixed points are shown as *solid lines* whereas unstable branches of fixed points are shown as *dotted lines*.

Bifurcation diagrams for linear differential equations are powerful teaching tools for describing what can and will happen in most basic differential equation systems with bifurcation parameters.

1.4 Exploring Higher Dimensional Stability

In the examples we have covered so far, we have only introduced *points* as potential steady/unsteady equilibrium states. If we move into systems with two or more state variables (equivalent to additional dimensions in the phase plane), we can extend this definition of equilibrium to other geometries, such as lines and even surfaces!

Consider the following system:

$$\dot{x} = 3y^2 - x$$

$$\dot{y} = x^2y^2 - y$$

At equilibrium, $\dot{x} = 0$ and $\dot{y} = 0$, so

$$x = 3y^2$$

$$y = y^2x^2$$

$$\implies x = 3y^2 \implies y^2 = \frac{x}{3}$$

$$y = y^2x^2 \implies y = \frac{x}{3}x^2 \implies y^*(x^*) = \frac{x^{*3}}{3}$$

Here we do not have fixed points, but rather, fixed curves! In this case, anything living on the curve $y^*(x^*)$ will be stable. This example is simply a curve, but it's not difficult to imagine a system where the equilibrium geometry is in the form of something intuitive ($x^2 + y^2 = c$, i.e. a circle). Incidentally, this happens so often in studies of dynamical systems that the special name of **limit cycles** have been given to systems with similar functional equilibria.

Other combinations of coupled system can give way to generalised closed loops, instead of perfect circles or ellipses. When a closed loop is found in the phase plane of a differential equation, **special properties of its respective solutions** should be noted.

Solutions starting either outside or inside of a *limit cycle* may rest along the stable geometry in phase space or diverge. Likewise, solutions starting outside of the limit cycles could either diverge or eventually come towards the limit cycle (shown in Figure 5.2).

This concept may be applied to familiar coordinate systems, such as: cartesian, cylindrical, spherical and beyond. Here, a cycle may be a surface with solutions free to roam inside of it or outside of it, *but not through it*!

Indeed, the **Poincaré-Bendixson theorem**[3] states that any orbit (a solution path) which stays in a compact region in 2D phase space approaches either a fixed point or a periodic orbit[4]. In addition to this concept, the theorem makes the claim (not proven here) that, **any n-dimensional system of differential equations will have an n-dimensional manifold representing its stable or unstable set of orbits**.

[3]Poincaré, H. (1892), "Sur les courbes définies par une équations différentielle"

[4]The Poincaré-Bendixson theorem also states that aperiodic behaviour can only occur in systems with 3 or more dimensions in continuous systems.

2

Applications of Continuous Differential Systems

The **effects of chaos** are visible in this phase space. By attempting to cover every point on the manifold, the system no longer has a logical, periodic path to follow. It attempts to be everywhere with no regard for a reversible path. Reversibility is a property of periodic dynamical systems. The aperiodicity of deterministic differential equations gives no regard to this concept.

2.1 Lorenz Equations and Rayleigh-Bénard Convection

Originally designed to try to predict the periodicity of weather patterns, the Lorenz equations are ultimately the resulting product of many years of work by multiple thinkers.

$$\frac{dX}{d\tau} = \sigma(Y - X) \tag{2.1}$$

$$\frac{dY}{d\tau} = rX - Y - XZ \tag{2.2}$$

$$\frac{dZ}{d\tau} = XY - bZ \tag{2.3}$$

In 1916, Lord Rayleigh tried to put into mathematics a phenomenon described by Henri Bénard and himself known as Rayleigh-Bénard convection. Rayleigh took the Navier-Stokes equations of fluid mechanics and found a steady-state solution of a **simple temperature gradient** throughout a fluid of depth H and temperature difference ΔT. Any deviation from this steady-state solution Rayleigh simply called *unstable*, and his work on the problem ended there.

Dr. Barry Saltzman revisited this problem in 1962 and tried to peer into the *behavior of all solutions, stable and unstable alike.* **Keeping the nonlinear terms** of the Navier-Stokes equations of fluid mechanics, Saltzman used his math background to convert the complicated system of nonlinear, partial differential equations to a set of nonlinear, ordinary differential equations using a technique known as the **Galerkin method**. The creative credit for this goes to Boris Galerkin, the Russian mathematician behind this technique.

The Galerkin method is well-known in the applied math community as a method of converting a continuous operator problem to a discrete problem. Today we consider the "Finite Element Method (**FEM**)" of numerical computing as a product of the Galerkin method. This differs from the "Finite Difference Method (**FDM**)" in that ***the FDM is an approximation to the differential equation and the FEM is an approximation to its solution***.

Although Saltzman derived the original set of ordinary differential equations shown in the beginning of this section, Ed Lorenz in 1963 decided to explore it further. Lorenz, a meterologist by training, numerically simulated the equations on his office computer in an attempt to understand weather system periodicity. He was naturally surprised to find that even by tweaking the initial conditions of the differential equations in the slightest, did he witness profound differences in solutions. Initially thinking this was an error on his Royal McBee LGP-30 computer, Lorenz explored this further, playing with values of the parameters and initial conditions.

Lorenz placed his findings in a paper titled *"Deterministic Nonperiodic Flow"*, published in 1963. Using **both linearisation theory and numerical methods on the nonlinear Lorenz equations**, Lorenz explored the behaviour of the eigenvalues and finds a dependence of parameters upon the system's behaviour.

Defined below are relevant descriptions of the parameters studied:

- σ = Prandtl number = Ratio of momentum diffusivity (kinematic viscosity) and thermal diffusivity

- r = normalised Rayleigh number = $\frac{Ra}{Ra_c}$ = Ratio of Rayleigh number and critical

Rayleigh number (Ra_c), which is the Rayleigh number defined by the offset of convection

- b = $\frac{4}{1+a^2}$ = Normalised aspect ratio of pre-defined[1] two-dimensional convective system

2.1.1 Stability Analysis

A cleaner representation of the Lorenz equations is:

$$\dot{x} = \sigma(y - x) \tag{2.4}$$

$$\dot{y} = rx - y - xz \tag{2.5}$$

$$\dot{z} = xy - bz \tag{2.6}$$

When seeking this system's fixed[2] points (denoted by x^*, y^*, z^*), we note that $x^* = y^* = z^* = 0$ is a trivial solution of the problem. To find the less obvious points, we set each equation in the ensemble equal to zero, a method described in our introductory section. Simple algebra illustrates the process:

$$\dot{x} = 0 \Longrightarrow \cancel{\sigma} y = \cancel{\sigma} x$$

[1]As defined in Saltzman's 1962 paper, *Finite Amplitude Free Convection as an Initial Value Problem*

[2]coordinate points in space such that $\dot{x} = \dot{y} = \dot{z} = 0$

$$\dot{y} = 0 \implies rx - xz = +y$$

$$\dot{z} = 0 \implies xy = bz$$

At equilibrium...

$$x = y$$

$$y = rx - xz$$

$$xy = bz$$

Since $x = y$ and because these equations are coupled...

$$\implies x = rx - xz \implies x = x(r - z) \implies 1 = r - z$$

$$\implies x^2 = bz$$

Rearranging...

$$\implies z = r - 1$$

Substituting...

$$\implies x^2 = b(r - 1)$$

We finally have $x^* = \pm\sqrt{b(r-1)}$. And since $x = y$, we can say $x^* = y^*$. We also collect the independent equation for z and add it to our repertoire to introduce the grand ensemble of equilibria:

$$x^* = \pm\sqrt{b(r-1)}$$

$$y^* = \pm\sqrt{b(r-1)}$$

$$z^* = r - 1$$

One should immediately see that due to the magnitudes of the parameters b and r, the locations of the fixed points in phase space **depend on given values of the parameters**. One may also observe that the parameter σ was quickly eliminated, illustrating the **independence of the Prandtl number in the determination of equilibria in the full Lorenz system**. Now that we have all[3] the equilibria points (steady and unsteady alike, thanks to the analytic method above), we wish to categorise them further.

To analyse the Lorenz system in this more advanced manner, it is useful to derive the Jacobian matrix of partial differentials for the Lorenz equations. The general form[4] of a Jacobian matrix ($\mathcal{J}$) in three dimensions is:

[3]Aside from the trivial solution. In the language of differential equations, the "trivial solution" is jargon for x=0, y=0, z=0, where the Lorenz system is trivially at equilibrium.

[4]where $\dot{x}_{i_{x_j}}$ denotes the derivative of the i^{th} equation with respect to the j^{th} variable

$$\mathcal{J} = \begin{pmatrix} \dot{x}_{1x_1} & \dot{x}_{1x_2} & \dot{x}_{1x_3} \\ \dot{x}_{2x_1} & \dot{x}_{2x_2} & \dot{x}_{2x_3} \\ \dot{x}_{3x_1} & \dot{x}_{3x_2} & \dot{x}_{3x_3} \end{pmatrix}$$

So in the case of the Lorenz system, the Jacobian matrix ($\mathcal{J}$) may be written more simply as...

$$\mathcal{J} = \begin{pmatrix} \dot{x}_x & \dot{x}_y & \dot{x}_z \\ \dot{y}_x & \dot{y}_y & \dot{y}_z \\ \dot{z}_x & \dot{z}_y & \dot{z}_z \end{pmatrix} \Longrightarrow \mathcal{J} = \begin{pmatrix} -\sigma & \sigma & 0 \\ (r-z) & -1 & -x \\ y & x & -b \end{pmatrix}$$

We recall our three equilibrium points and establish notation for each separate Jacobian matrix ($\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$), corresponding to each separate fixed point. In addition, as the reader will see, we will set $\Gamma = \sqrt{b(r-1)}$ to simplify our analysis.

$$\mathcal{J}_1 = \mathcal{J}_{(0,0,0)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$$

$$\mathcal{J}_2 = \mathcal{J}_{(\Gamma,\Gamma,r-1)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\Gamma \\ \Gamma & \Gamma & -b \end{pmatrix}$$

$$\mathcal{J}_3 = \mathcal{J}_{(-\Gamma,-\Gamma,r-1)} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \Gamma \\ -\Gamma & -\Gamma & -b \end{pmatrix}$$

As performed in Section 1.2 of this thesis, we seek to establish the behaviour of the eigenvalues specific to each Jacobian matrix ($\mathcal{J}_1$, $\mathcal{J}_2$, $\mathcal{J}_3$), and thus, repeat the mathematics:

$$|\mathcal{J}_1 - I\lambda_1| = 0 \to \begin{vmatrix} -\sigma-\lambda_1 & \sigma & 0 \\ r & -1-\lambda_1 & 0 \\ 0 & 0 & -b-\lambda_1 \end{vmatrix} = 0$$

$$|\mathcal{J}_2 - I\lambda_2| = 0 \to \begin{vmatrix} -\sigma-\lambda_2 & \sigma & 0 \\ 1 & -1-\lambda_2 & -\Gamma \\ \Gamma & \Gamma & -b-\lambda_2 \end{vmatrix} = 0$$

$$|\mathcal{J}_3 - I\lambda_3| = 0 \to \begin{vmatrix} -\sigma-\lambda_3 & \sigma & 0 \\ 1 & -1-\lambda_3 & \Gamma \\ -\Gamma & -\Gamma & -b-\lambda_3 \end{vmatrix} = 0$$

Characteristic equation for $\mathcal{J}_1$:

$$\lambda_1{}^3 + \lambda_1{}^2(\sigma + b + 1) + \lambda_1(b + \sigma + \sigma b + \sigma r) - \sigma r b + \sigma b = 0$$

Characteristic equation for $\mathcal{J}_2$:

$$\lambda_2{}^3 + \lambda_2{}^2(\sigma + b + 1) + \lambda_2(\Gamma^2 + b + \sigma b) + 2\sigma\Gamma^2 = 0$$

Characteristic equation for $\mathcal{J}_3$:

$$\lambda_3{}^3 + \lambda_3{}^2(\sigma + b + 1) + \lambda_3(\Gamma^2 + b + \sigma b) + 2\sigma\Gamma^2 = 0$$

These equations would not be easily solvable without the advent of computerised symbolic algebraic manipulation. When solved with Wolfram's Mathematica, we see mathematical results that verify the physical state. If $r < 1$, then the solution decays to the center of the phase plane to a fixed point. This represents the non-convective state.

If $r > 1$, two of the eigenvalues appear as complex numbers. Complex numbers, in the field of eigenvalue analysis, mean that solutions have tendencies for oscillation between solutions. Indeed, once we enter the chaotic realm of the Lorenz equations, solutions span an area known as that Lorenz Attractor (as seen in Figure 5.6).

Unfortunately for the analytic purist, this is as far as we may go in manually assessing the properties of the Lorenz equations. Studies of group theory in mathematics have proven that it is not possible to solve for any polynomials past certain quintic (5th power) polynomials. What this means is that **these methods described here**

cannot be applied to coupled dynamical systems with more five sets of coupled linear or nonlinear differential equations.

However intelligent, motivated and dedicated to their trades, mathematicians and physicists from the earth 20th century did not have the luxury of computing power. This ability needed to be developed, fostered and understood. This was primarily the reason why Ed Lorenz was able to assess what he did. Standing on the shoulder of giants, Lorenz used a computer to place the final straw that broke the proverbial camel's back, revealing in its fallout the beginnings of numerical aperiodicity.

Lorenz goes on to show in his paper how limiting values of parameters cause eigenvalues to switch signs and thus rock **back and forth between stability and instability**. Indeed, by plotting a bifurcation diagram numerically, Lorenz was able to show that the system undergoes a **Hopf bifurcation** at a particular value of the normalised Rayleigh number r...

$$r = \frac{Ra}{Ra_c}$$

...where Ra is the Rayleigh number of the system and where Ra_c is the characteristic, critical Rayleigh number of the system, which is the Rayleigh number associated with the onset of heat convection.

In describing the equilibria of the Lorenz system in the previous section, we recall that

$$x^* = \pm\sqrt{b(r-1)}$$

$$y^* = \pm\sqrt{b(r-1)}$$

$$z^* = r - 1$$

So if $r < 1$, x^* and y^* aren't even real numbers, but rather complex[5] numbers. The implications of complex numbers in the scope of eigenvalues of a Jacobian matrix of a differential equation system are interesting, but we know that **aperiodic convective behaviour may only arise if** $r > 1$ (i.e. $Ra > Ra_c$).

In a strictly physical sense, the value of the normalised Rayleigh number (r) will grow in response to a growing temperature gradient. Ed Lorenz also discovered an obstacle due to the growth of this parameter. One of the last formulas in his paper (derived from eigenvalue analysis of the Jacobian matrix) is given by

$$r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$$

This formula states that for any given aspect ratio b and any given fluid with Prandtl number σ (where the material property is now relevant), that **the critical**

[5]Due to the behaviour of having a negative value within the square root function in both x^* and y^*.

Hopf bifurcation value (r_H) for r is now known! This is a natural consequence of the fact that aperiodicity is dependent on the physical dimensions of an environment.

As long as $\sigma > b + 1$, steady convection is ***unstable*** for sufficiently high Rayleigh numbers. What we've been building up to in this entire discussion is that **we can actually force a physical system like this into an unstable regime in order to witness how it transitions from a state of stability to instability just by *turning up the heat***. Lorenz stated that chaos would be apparent if the system reached a critical value of r, which he found to follow the equation above for a given σ and given b, provided $\sigma > b + 1$.

When Lorenz used σ=10 and b=8/3, the equation noted above, yielded $r = 470/19 \approx 24.74$. So any value of r greater than this could very well make the entire system aperiodic in X,Y, and Z variables. This is what is represented in the famous plots he produced, known as the *Lorenz butterfly*, shown in Figure 5.6.

Putting things into perspective, if we were to actually have a slab in the geometry of something portable like a fish tank, **we could simulate this ourselves**, provided we had a heating and cooling source (see Figure 6.7). The nature of aperiodicity is a very visual one and many researchers take pride in being able to visually show the results of their numerical simulations. Numerically and/or experimentally reproducing Rayleigh-Bénard convection is *a very rewarding effort* if one decides to put the time and energy into performing such a task.

Reviewing the parameters...

- σ = Prandtl number (a material property). Experimenter can be more selective in choosing a fluid that adheres to a desired systemic behaviour.

- b = $\frac{4}{1+a^2}$ = Normalised aspect ratio. (Determined by the dimensions of the physical system)

- r = $\frac{Ra}{Ra_c}$ = Heat source optimisation (How severely the temperature differential of system can be manipulated)

The Lorenz system shows that mathematics and physics can truly come together to create intriguing and potentially useful results. We use the tools from bifurcation theory in explaining why we have a **transition from laminar convection to turbulent convection**, a physical phenomenon. The power of this study comes from applying these simple models to larger, physical systems. The applicator must retain the techniques of the results, but also keep in mind that larger systems are proportionally more complicated.

3

Magnetohydrodynamics

3.1 Maxwell's Equations & Fluidic Force

In introducing the Lorenz equations and the concept of dynamical systems from the perspective of applied mathematics, a seemingly natural convergence may be apparent. When looking at the equations of electromagnetic properties for a fluid via Maxwell's equations (presented below), there may be times when the evolution of a thermally convective state, is desired.

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\vec{\nabla} \times \vec{B} = \mu \vec{J}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_o}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{F} = q[\vec{E} + (\vec{\xi} \times \vec{B})]$$

From the perspective of vectorial Newtonian mechanics, the last equation translates the four coupled differential equations above it, into an understandable entity. The **electromagnetic force equation** shows how an electric field $\vec{E}$ interacts with a magnetic field $\vec{B}$, to produce a force $\vec{F}$, provided there exists a velocity of fluid flow $(\vec{\xi})$ in physical space, containing particles with charge q. This generalised equation also applies for instances with no electric field present $(\vec{E} = 0)$ or instances with either no fluid field velocity $(\vec{\xi} = 0)$ or no magnetic field $(\vec{B} = 0)$. In the most trivial case, if there are no charges (q) present, the force contributed by the electromagnetic force is exactly zero.

It is logical to conclude that superposition plays a role in describing this combination. It is also important to note, as it is not trivial, that these equations do not simply contradict each other or cancel each other out. In sections to come, we see how previous thinkers have confronted this issue, and how we can try to extend the power of the Lorenz equations to obtain systematic behaviour given mathematical machinery we already have.

3.2 Ferroconvection

In studying Rayleigh - Benard convection, Barry Saltzman started with an infrastructure of governing equations for his problem. He knew that fluid flow would require the Navier-Stokes equations and that any introduction of a thermal gradient would naturally introduce Newton's Heat equation. The subtle beauty of Saltzman's derivation was his introduction of a non-linear mixing term, which brought convection into the picture in the first place. It is this nonlinear behaviour which makes aperiodicity so stark and apparent, lending itself to result in an infinite series of trigonometric convective modes, as shown by Saltzman.

What I wish to do now with this thesis is give the reader a taste of what might occur if we went back to Saltzman's original set of equations and introduced one more equation to the ensemble. Given Saltzman's fluid and thermodynamic equations, it can be very easily seen that the fluid can be a ferromagnetic one (such as mercury or salt water) which obeys both fluid and thermodynamic equations. In addition to this, a ferrofluid's magnetic natures gives it a third master: the Maxwellian electromagnetic equations.

Would it not, then, be possible to revisit Saltzman's derivation and add this additional physical property into the mathematical mix?

3.2.1 Expanding on Saltzman's Technique

The following definitions correspond to Saltzman's *"Finite Amplitude Free Convection as an Initial Value Problem - I"*:

- x,y = horizontal coordinates
- z = vertical coordinate
- t = time (unscaled)
- ρ = density
- p = pressure
- T = temperature
- ϵ = coefficient of volume expansion
- κ = coefficient of thermal diffusivity
- ν = kinematic viscosity
- Ψ = streamfunction

We know that within the Oberbeck-Boussinesq approximation, fluid flow is taken to be incompressible. In addition, Saltzman states that *"the problem is simplified by constraining convective rolls to develop in the x-z plane ($v = \partial/\partial y = 0$)"*, assuming the problem is situated in a standard 3-D cartesian coordinate system.

The governing equations are...

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{\partial P}{\partial x} - \nu\nabla^2 u = 0$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{\partial P}{\partial z} - g\epsilon T_1 - \nu\nabla^2 w = 0$$

$$\frac{\partial T_1}{\partial t} + u\frac{\partial T_1}{\partial x} + w\frac{\partial T_1}{\partial z} - \kappa\nabla^2 T_1 = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$u = -\frac{\partial \psi}{\partial z}, w = \frac{\partial \psi}{\partial x}$$

When the vorticity ($\omega = \nabla\psi^2$, where commonly $\vec{\omega} = \vec{\nabla} \times \vec{\xi}$) of the fluid flow is taken into account, instead of direct cartesian velocities, some simplifications may be made. The final equations of Saltzman are

$$\frac{\partial}{\partial t}\nabla^2\psi - \frac{\partial \psi}{\partial z}\frac{\partial}{\partial x}\nabla^2\psi + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial z}\nabla^2\psi - g\epsilon\frac{\partial \theta}{\partial x} - \nu\nabla^4\psi = 0$$

$$\frac{\partial \theta}{\partial t} - \frac{\partial \psi}{\partial z}\frac{\partial \theta}{\partial x} + \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial z} - \frac{\Delta T_o}{H}\frac{\partial \psi}{\partial x} - \kappa\nabla^2\theta = 0$$

where, $\nabla^4\psi = \nabla^2\nabla^2\psi = \partial^4\psi/\partial x^4 + \partial^4\psi/\partial z^4 + 2\frac{\partial^4\psi}{\partial x^2 z^2}$. ∇^4 is known as the biharmonic operator.

Saltzman then states: *"We can introduce a further notational simplication by writing the non-linear advective terms in the form of a Jacobian operator"*

$$\frac{\partial(a,b)}{\partial(x,z)} = \left(\frac{\partial a}{\partial x}\frac{\partial b}{\partial z} - \frac{\partial b}{\partial x}\frac{\partial a}{\partial z}\right)$$

Finally giving us:

$$\frac{\partial}{\partial t}\nabla^2\psi - \frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)} - g\epsilon\frac{\partial\theta}{\partial x} - \nu\nabla^4\psi = 0$$

$$\frac{\partial\theta}{\partial t} - \frac{\partial(\psi,\theta)}{\partial(x,z)} - \frac{\Delta T_o}{H}\frac{\partial\psi}{\partial x} - \kappa\nabla^2\theta = 0$$

$$\frac{dT}{dt} = \kappa\nabla^2 T$$

Lorenz then continues where Saltzman leaves off by expanding these equations using double Fourier series. Lorenz takes the highest order terms of this expansion and uses them to construct the terms in the system of three non-linear differential equations.

3.3 Magnetic Induction equation

The **magnetic induction equation** is a consequence of the mathematical manipulation of Maxwell's Equations. Shown below, the form exists in nature, and is thus expressed in three spatial coordinates and one time coordinate components.

$$\frac{d\vec{B}(x,y,z,t)}{dt} = \vec{\nabla} \times (\vec{\xi} \times \vec{B_o}) + \eta \nabla^2 \vec{B_o} \tag{3.1}$$

What makes this equation intriguing is that as long as there is a continuous flow of fluid velocity ($\vec{\xi}$) and an ambiant magnetic field ($\vec{B_o}$), either due to a current in a wire or due to a magnet, there will be a change in the system-wide magnetic field. By Maxwell's Equations, this will generate a $\vec{\nabla} \times \vec{E}$ effect. This phenomenon in turn will provide an induced electric field which contributes to the system's magnetic field in the first place. Since there is no such thing as free energy, we realize that we need a source of velocity in the first place, to guarantee this can occur. Fortunately, we recall that thermal convection gives way to fluid velocity fluctuations.

3.4 Self-Sustaining Flows

Self-sustaining means that if heat and magnetic energy is sufficient (contributed by a thermal gradient, which in turn sustains the system's magnetic field), then we can have a chaotic flow in a ferrofluid. The reader should combine what he or she has already read about thermal convection to see how this is true. **Kinetic dynamo theory** is used to approximate some ideal flows of this type.

$$\frac{d\vec{B}}{dt} = \vec{\nabla} \times (\vec{\xi} \times \vec{B_o}) + \eta \nabla^2 \vec{B_o} \tag{3.2}$$

The equation above states that a velocity field $\vec{\xi}$ will cause a change in a magnetic field due to the advection of the fluid with a previously present magnetic field $\vec{B_o}$. In addition, when $\vec{\xi} = \vec{0}$, the equation is reduced to

$$\frac{d\vec{B}}{dt} = \eta \nabla^2 \vec{B}_o$$

which is very much analogous to the diffusion equation for heat, if the operator is acting on real, physical field. The Laplacian operator is responsible for the dissipation of a field's energy.

The more mathematically complicated term $[\vec{\nabla} \times (\vec{u} \times \vec{B})]$ was studied considerably by Alfen in the early 20th century. His work is manifested in what is now known as **Alfen's Theorems**. The background required to properly approach this aspect of the magnetic induction equation is beyond the scope of this thesis and will be studied at a future date.

4

Conclusions

This thesis has covered the analytical consequences of including terms to the Lorenz equations. However insightful, I feel that a more in-depth look at these concepts from the vantage point of numerical simulations is necessary.

What was once an impossible task is now possible. Thanks to the advent of computing and even faster processing for complex problems, computational physics and fluid mechanics go hand-in-hand. The interpretation of both complex analytic equations and intricate numerical codes by human researchers is necessary to achieve these goals.

5

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To the aforementioned, on behalf of your students, past and present, I thank you. We would be lesser people without you.

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List of Figures

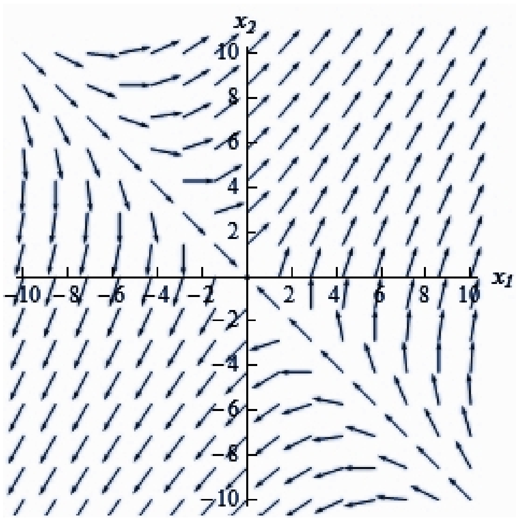

Figure 5.1: Phase Plane for Arbitrary System

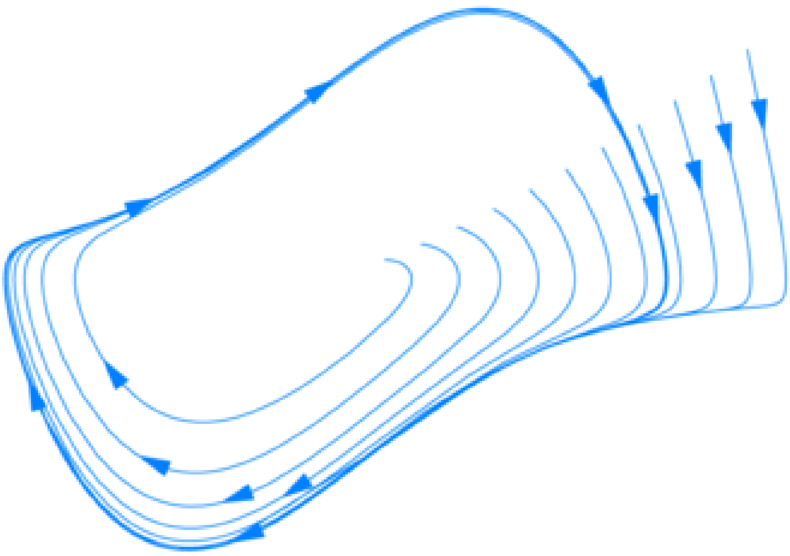

Figure 5.2: Example of a Limit Cycle

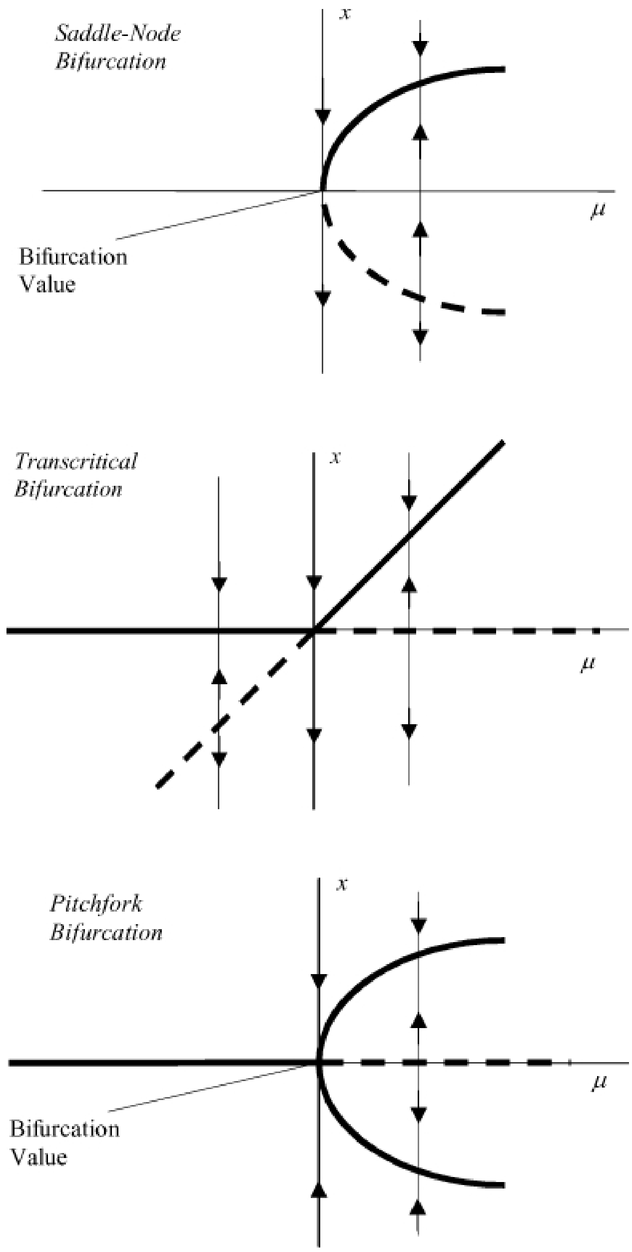

Figure 5.3: Various Types of Bifurcations

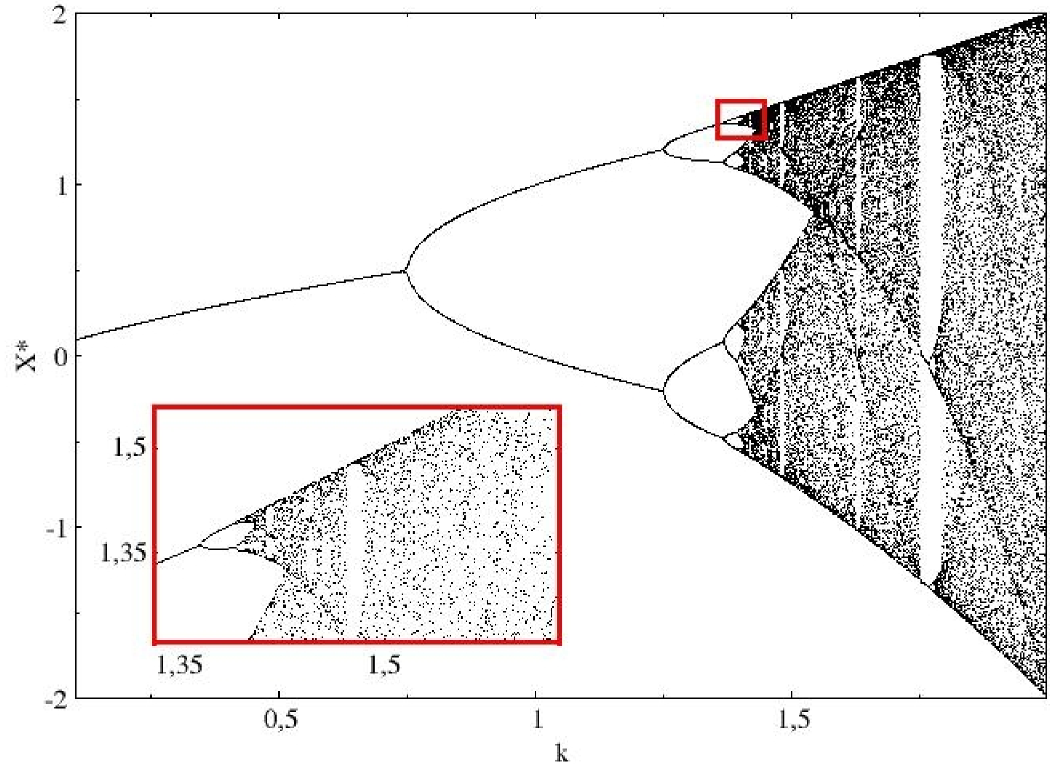

Figure 5.4: Zooming in on Ambiguity

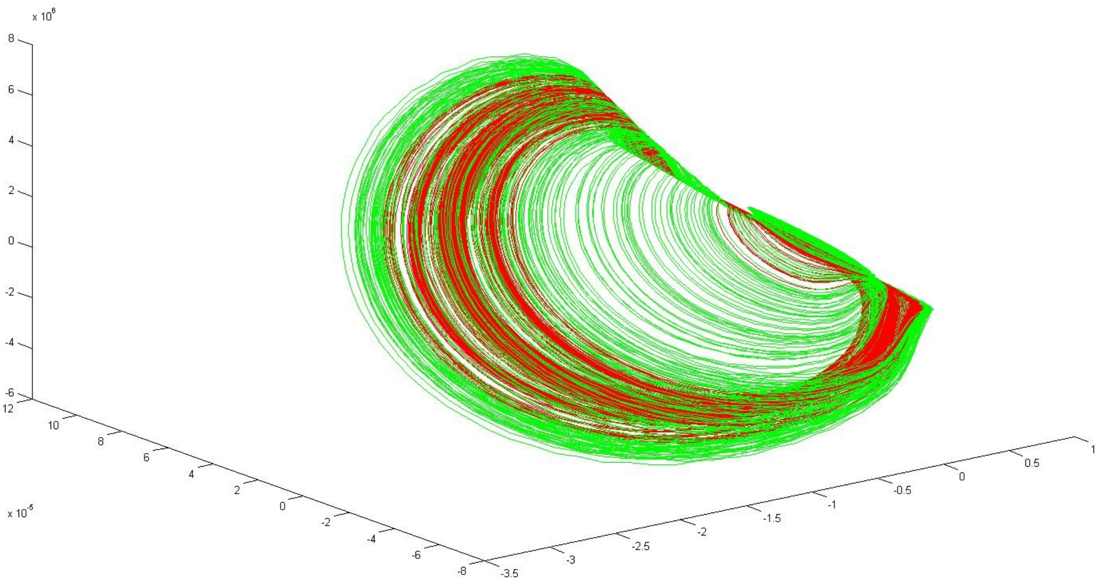

Figure 5.5: Juxtapositioning chaotic solutions (lighter) and limit cycles (darker)

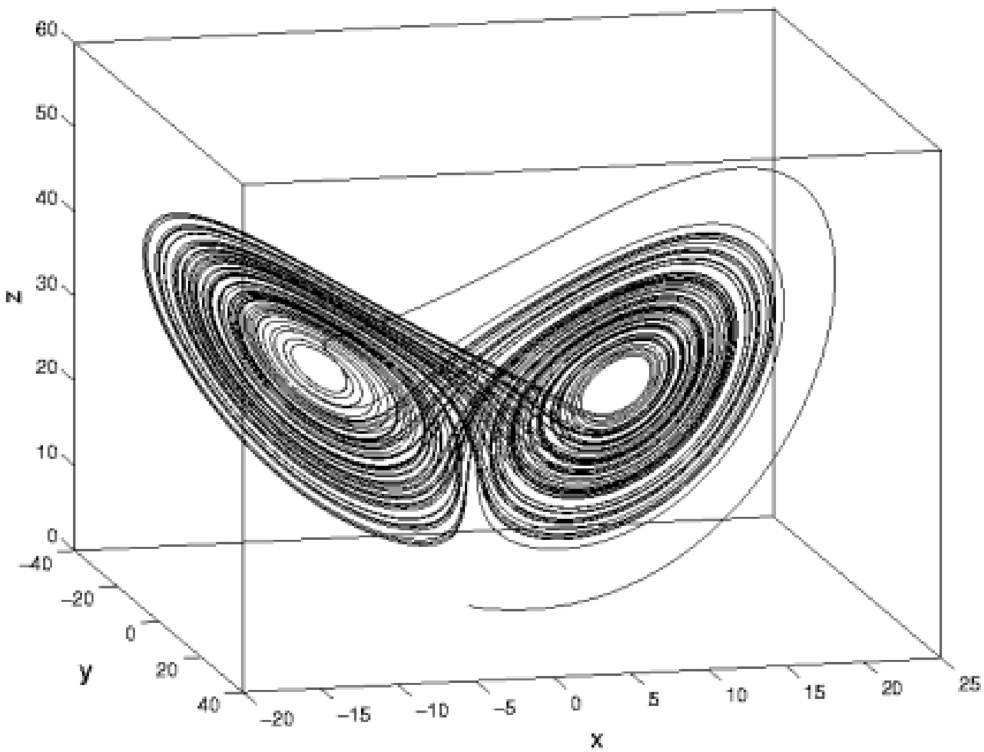

Figure 5.6: The Lorenz Attractor

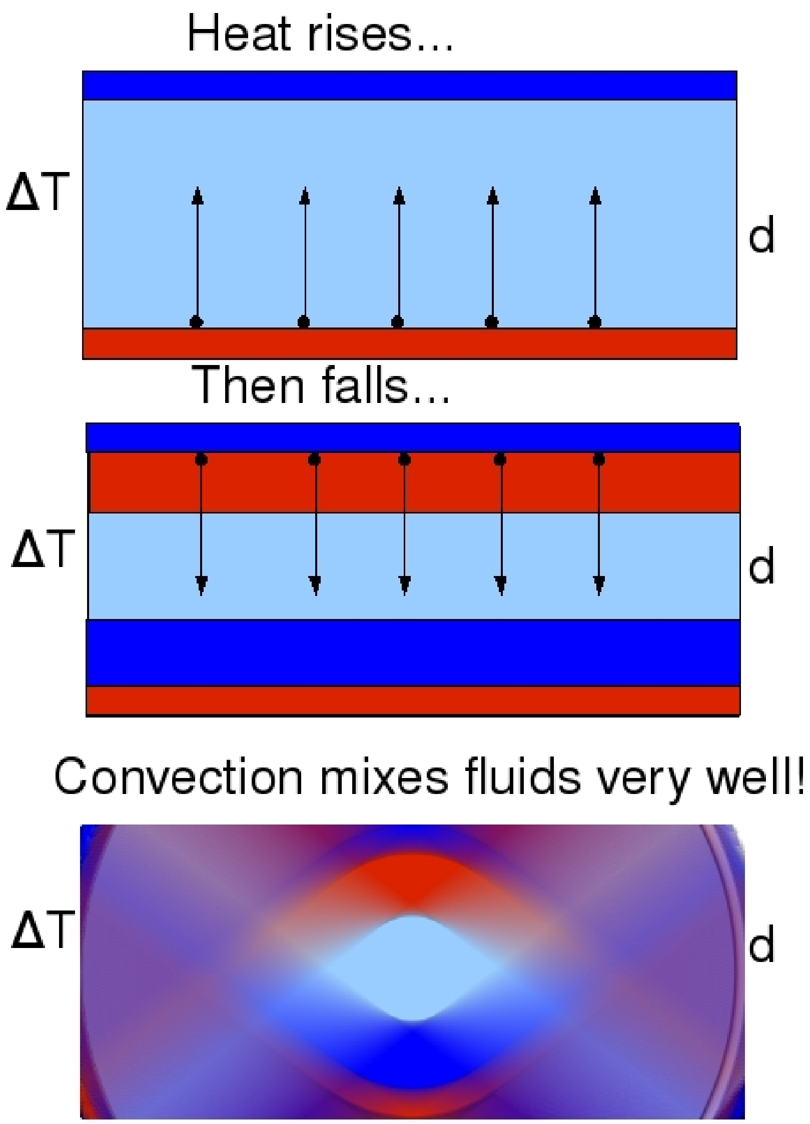

Figure 5.7: Rayleigh Bénard Convection